Exam Linear Algebra 1
31 January 2013
- solutions -

1. (a) $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{pmatrix} \xrightarrow[(iii)-4(i)\to(iii)]{(ii)-2(i)\to(ii)} \begin{pmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -9 \end{pmatrix}$

$\xrightarrow{(iii)-2(ii)\to(iii)} \begin{pmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow[-\frac{1}{3}(iii)\to(iii)]{-\frac{1}{3}(ii)\to(ii)} \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ [4]

(b) rank $A = 3$ [2]

(c) $A\underset{\sim}{x} = \underset{\sim}{0}$ has only solution $\underset{\sim}{x} = 0$ (consider row echelon form) [4]

(d) augmented matrix

$(A|\underset{\sim}{b}) = \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right) \xrightarrow[\text{above}]{\text{as}} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

$\Rightarrow x_3 = 3,\ x_2 = -1 - x_3 = -4,\ x_1 = 1 - 4x_2 - 2x_3 = 11$

$\Rightarrow$ solution set $= \left\{ \begin{pmatrix} 11 \\ -4 \\ 3 \end{pmatrix} \right\}$ [4]

2. (a) Let $a_1, \dots, a_n$ be the column vectors of A.

Then $b \in R(A) \iff$ there exists $x \in \mathbb{R}^n$ such that

$x_1 a_1 + \dots + x_n a_n = b$

$\iff$ there exists $x \in \mathbb{R}^n$ such that $Ax = b$

$\iff$ $Ax = b$ has solution $x \in \mathbb{R}^n$ [4]

(b) $N(A) = \{0\} \iff$ $Ax = 0$ has unique solution $x = 0$

$\iff$ $x_1 a_1 + \dots + x_n a_n = 0$ has unique solution $x_1 = \dots = x_n = 0$

$\iff$ $a_1, \dots, a_n$ are linearly independent [4]

(c) Let $b \in R(A)$

$Ax = b$ has more than two solutions.

"$\Leftarrow$": Suppose $Ax = b$ has more than two solutions

$\Rightarrow$ there exist $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \neq x_2$ such

that $Ax_1 = b$ and $Ax_2 = b$ [3]

$\Rightarrow$ $A(x_1 - x_2) = 0$ with $x_1 \neq x_2$

$\Rightarrow$ column vectors of A not lin. independent

"$\Rightarrow$": Suppose column vectors of A linearly dependent

$\Rightarrow$ $Ax = 0$ has solution $y \neq 0$

$\Rightarrow$ $Ax = b$ has besides solution $x \in \mathbb{R}^n$ [3]

(which exists since $b \in R(A)$) also

solution $x + y$ since $A(x + y)$

$= Ax + Ay = b + 0 = b$ where $y \neq 0$

$\Rightarrow$ $Ax = b$ has more than one solution.

3. (a) $A X B = A + B$

multiplication from the left by A^{-1} and from the right by B^{-1} yields [4]

$$X = A^{-1}(A+B)B^{-1} = B^{-1} + A^{-1}$$

(b) $(AB)^T = B^T A^T$ (always)

however not $AB = BA$ hence $(AB)^T = (BA)^T = A^T B^T$ [4]

(c) A invertible => there exists a matrix B such that

$$AB = BA = I$$

Taking the transpose on both sides yields

$$B^T A^T = A^T B^T = I.$$

Hence $B^T = (A^T)^{-1}$ where $B = A^{-1}$. [4]

[3]

4. (a) P_n has basis $\{1, x, \ldots, x^{n-1}\}$ [2]

(b) $\dim P_n = n$

(c) let $p, q \in P_4$, $\alpha, \beta \in \mathbb{R}$

$$\Rightarrow T(\alpha p(x) + \beta q(x)) = (\alpha p(x) + \beta q(x)) - \tfrac{1}{2}x^2(\alpha p(x) + \beta q(x))''$$

$$= \alpha p(x) + \beta q(x) - \alpha \tfrac{1}{2}x^2 p''(x) - \beta \tfrac{1}{2}x^2 q''(x)$$

$$= \alpha\left(p(x) - \tfrac{1}{2}x^2 p''(x)\right) + \beta\left(q(x) - \tfrac{1}{2}x^2 q''(x)\right)$$

$$= \alpha T(p(x)) + \beta T(q(x))$$

[4]

(d) $p \in \ker T \Leftrightarrow T(p(x)) = 0$

$\Leftrightarrow p(x) - \frac{1}{2}x^2 p''(x) = 0$

Let $p(x) = a + bx + cx^2 + dx^3$ with $a,b,c,d \in \mathbb{R}$

$\Rightarrow p(x) - \frac{1}{2}x^2 p''(x) = a + bx + cx^2 + dx^3 - \frac{1}{2}(2cx^2 + 6dx^3)$

$= a + bx - 2dx^3 = 0$

$\Leftrightarrow a = b = d = 0$ and $c \in \mathbb{R}$

[4]

$\Rightarrow \ker T = \operatorname{span}\{x^2\}$

(e) From above we see

$T(a + bx + cx^2 + dx^3) = a + bx - 2dx^3$

$\Leftrightarrow [T]_E \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \\ -2d \end{pmatrix}$ for all a, b, c, d

applying $[T]_E$ to e_k, $k = 1,2,3,4$, yields the k^{th} column vector of $[T]_E$.

Hence $[T]_E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

[4]

(f) $\operatorname{rank}[T]_E = \operatorname{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (row echelon form of $[T]_E$)

$= 3$

[3]

5. (a) Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in W^\perp$ and $\alpha, \beta \in \mathbb{R}$

$\Rightarrow$ Let $\underset{\sim}{y} \in W$. Then

$$(\alpha \underset{\sim}{x}_1 + \beta \underset{\sim}{x}_2)^T \underset{\sim}{y} = \alpha \underset{\sim}{x}_1^T \underset{\sim}{y} + \beta \underset{\sim}{x}_2^T \underset{\sim}{y}$$

$$= 0$$

[4]

$$\Rightarrow \quad \alpha \underset{\sim}{x}_1 + \beta \underset{\sim}{x}_2 \in W^\perp$$

Moreover $W^\perp \neq \emptyset$ since $\underset{\sim}{0} \in W^\perp$

(b) $\underset{\sim}{x} \in W \cap W^\perp \Leftrightarrow \underset{\sim}{x}^T \underset{\sim}{x} = 0$

$$\Leftrightarrow \sum_{k=1}^{n} x_k^2 = 0$$

[2]

$$\Leftrightarrow \underset{\sim}{x} = \underset{\sim}{0}$$

(c) $\underset{\sim}{x} \in N(A^T) \Leftrightarrow x_1 \vec{a}_1^T + \dots + x_n \vec{a}_n^T = \underset{\sim}{0}$

where $\vec{a}_1, \dots, \vec{a}_n$ are the row vectors of A

$$\Leftrightarrow \sum_{k=1}^{n} x_k A_{ki} = 0 \quad \text{for all } i = 1, \dots, n$$

$$\Leftrightarrow \underset{\sim}{x}^T \underset{\sim}{a}_i = 0 \quad \text{— " —}$$

$$\Leftrightarrow \underset{\sim}{x} \in R(A)^\perp$$

[8]

6. (a) $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{vmatrix}$

$= (1-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 5 & -1-\lambda \end{vmatrix}$

$= (1-\lambda)\left((3-\lambda)(-1-\lambda) - 5\right)$ [3]

(b) $p(\lambda) = 0$

$\Leftrightarrow \lambda = 1$ or $(3-\lambda)(1-\lambda) = 5$ [3]

$\Leftrightarrow \lambda = 1$ or $\lambda = 4$ or $\lambda = -2$

(c) eigenvectors for eigenvalue λ are the nontrivial solutions of $(A - \lambda I)\underset{\sim}{v} = 0$

$\lambda_1 = 1$: $\begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{pmatrix} \underset{\sim}{v} = 0$

$\Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{pmatrix} \underset{\sim}{v} = 0 \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 9 & 0 \end{pmatrix} \underset{\sim}{v} = 0$

eigenvector $\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

similarly: for $\lambda_2 = 4$, $\underset{\sim}{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ [6]

$\lambda_3 = -2$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$

(d) A has 3 eigenvectors. Hence A is diagonalizable. Set $T = \begin{pmatrix} \underset{\sim}{v}_1 & \underset{\sim}{v}_2 & \underset{\sim}{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -5 \end{pmatrix}$

$\Rightarrow T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ [4]